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THE PROSPER GENERAL CIRCULATION MODEL, AN ADAPTED FORM OF THE SANDIA OCEAN MODELLING METHOD: A NUMERICAL DESCRIPTION

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SUMMARY

The PROSPER General Circulation Model (PGCM) is a three-dimensional model based on the incompressible Navier–Stokes equations, an equation of state and the heat equation. The hydrostatic approximation and the rigid lid approximation are used. The system of equations is converted into an equivalent form in which the surface pressure is more directly expressed in terms of a two-dimensional Poisson equation. The finite difference method is described and analysed. In particular, the iteration method within every time step to determine the new surface pressure and velocity components, and numerical diffusion aspects due to the use of the staggered Arakawa-C grid are looked at. Since part of the development of the PGCM code is a result of studying the Sandia Ocean Modelling System (SOMS), a comparison is made with respect to the concepts used in both models.

KEY WORDS Incompressible flow Hydrostatic approximation Pressure Poisson Equation Staggered grid Numerical diffusion

1. INTRODUCTION

The starting point for the PROSPER General Circulation Model (PGCM) discussed here is the Sandia Ocean Modelling System (SOMS) developed by Dietrich *et al.*[1] The performance of SOMS and a comparison with other models—in particular the Bryan–Semptner model[2,3]—in determining the large-scale circulation in the North Atlantic has been described by the same authors.[1] An application to determine the general circulation in Lake Neuchâtel (Switzerland) has been described by Zuur and Dietrich.[4] SOMS consists of two coupled models: the free-stream model and a bottom boundary layer model to describe the turbulent processes in the boundary layer. In this paper, references made to SOMS relate only to the free-stream part.

A specific and important aspect of SOMS is the predictor–corrector method within a time step to determine the new velocity components and surface pressure. In the case of Neumann boundary conditions at the surface and bottom of the domain the method used in PGCM is equivalent to the one used in SOMS. In the case of Dirichlet conditions, however, in PGCM an extended form of the predictor–corrector method is used. Vertical diffusion is, as in SOMS, treated implicitly. In contrast with SOMS, in PGCM the Coriolis term is taken explicitly in the numerical scheme. Although explicit treatment of this term implies an *a priori* limitation by the Coriolis parameter for the time step, for lake circulations and mesoscale processes in the ocean the time step is not so much limited by this parameter but by internal wave propagation and advection and horizontal diffusion terms. Therefore this disadvantage in comparison with SOMS is not relevant for these

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processes. The advantages of explicit treatment of the Coriolis term will become apparent in what follows.

In this paper a description of the PGCM numerical algorithm is presented. The iteration method within a time step to determine the new velocity components and surface pressure is discussed in detail. Criteria for stability are derived. An analysis of numerical diffusion aspects in both the SOMS and the PGCM code is performed.

2. THE CONTINUUM EQUATIONS

2.1 The basic equations

The circulation model is based on the Navier–Stokes equations written in primitive variables. Scaling estimates of the terms that appear in these equations justify the following approximations to simplify the resulting system.[5,6]

(i) The Bousinesq approximation: density variations are neglected in the horizontal momentum and mass conservation equations but are included when they are associated with buoyancy forces (in the vertical momentum equation). Density is assumed independent of pressure and varies only with temperature.
(ii) The hydrostatic approximation: all terms in the vertical momentum equation are neglected except the buoyancy and pressure gradient terms.
(iii) The rigid lid approximation.
(iv) Turbulence is represented by means of horizontal and vertical turbulent viscosity coefficients.

The turbulent viscosity coefficients are taken as constant. ρ is the normalized density (i.e. density divided by reference density). The kinematic pressure (i.e. pressure divided by reference density) is denoted by a capital P in order to distinguish it from the kinematic surface pressure p that will appear later on. The resulting continuum equations are stated below:

the horizontal momentum equations

$$u_t+\nabla\cdot(u\mathbf{v})-fv=-P_x+A_H\nabla_h^2 u+A_V u_{zz},$$
$$v_t+\nabla\cdot(v\mathbf{v})+fu=-P_y+A_H\nabla_h^2 v+A_V v_{zz};$$

the hydrostatic approximation equation

$$P_z=\rho g;$$

the mass conservation equation

$$u_x+v_y+w_z=0;$$

the equation of state

$$\rho=1-\beta(T-T_0);$$

the heat equation

$$T_t+\nabla\cdot(T\mathbf{v})=K_H\nabla_h^2 T+K_V T_{zz}.$$

The above equations apply to a bounded domain Ω for $t>0$, subjected to Dirichlet or Neumann boundary conditions at $\partial\Omega$ for the horizontal velocity components and temperature. Owing to the hydrostatic approximation, second-order spatial derivatives of the vertical velocity component w have disappeared. A consistent boundary condition for w is $wn_3=0$, in which n_3 is the third

component of the unit-outward pointing normal $\mathbf{n}=(n_1, n_2, n_3)$ at the boundary $\partial\Omega$. The initial conditions are

$$\mathbf{v}(x, 0)=\mathbf{v}_0(x) \qquad \text{in } \bar{\Omega}\equiv\Omega\cup\partial\Omega,$$

with

$$\nabla\cdot\mathbf{v}_0=0 \qquad \text{in } \Omega,$$
$$T(x, 0)=T_0(\mathbf{x}) \qquad \text{in } \bar{\Omega}.$$

The following symbols are used:

x, y, z	spatial co-ordinates in a left-handed Cartesian system (z-axis points downwards) [L]
t	time [T]
T	temperature [Θ]
u, v, w	velocity components ($\mathbf{v}=u, v, w$) [L T^{-1}]
P	kinematic pressure [L^2 T^2]
ρ	normalized density
g	effective gravitational acceleration [L T^{-2}]
A_H, A_V, K_H, K_V	turbulent viscosity and diffusion coefficients [L^2 T^{-1}]
f	Coriolis parameter [T^{-1}]
β	coefficient of thermal expansion [Θ^{-1}]
∇	$\equiv(\partial/\partial x, \partial/\partial y, \partial/\partial z)$
∇_h^2	$\equiv\partial^2/\partial x^2+\partial^2/\partial y^2$.

2.2 The equivalent formulation

Like the SOMS model, the PGCM model uses an equivalent form of the above set of equations based on *the equivalence assertion for incompressible flows* as described by Gresho and Sani.[7] This assertion, although not formally proved by the authors, has been verified thoroughly in practice, among others by Abdallah and Dreyer.[8] We will assume its correctness. The assertion is used to convert the original set of equations into one in which the pressure is more directly expressed in terms of the so-called pressure Poisson equation, which is derived from the mass conservation and momentum equations, and which will replace (together with an appropriate boundary condition) the mass conservation equation, giving the equivalent formulation of the problem. The assertion in its original form was stated with respect to the full incompressible Navier–Stokes equations. We apply the same ideas to the case in which the hydrostatic approximation is used.

Let us derive the equivalent formulation in the hydrostatic case. From the hydrostatic equation and the equation of state it follows that

$$P=p(x, y, t)+gz-g\beta\int_0^z T(x, y, \zeta, t)\mathrm{d}\zeta,$$

in which p is the surface pressure (which is different from the atmospheric pressure owing to the rigid lid approximation). Define the terms R and S, which are treated explicitly in the integration scheme:

$$R=g\beta\int_0^z T_x\mathrm{d}\zeta-\nabla\cdot(u\mathbf{v})+fv+A_H\nabla_h^2 u,$$

$$S=g\beta\int_0^z T_y\mathrm{d}\zeta-\nabla\cdot(v\mathbf{v})-fu+A_H\nabla_h^2 v.$$

With the above definitions the momentum equations take the form

$$u_t = -p_x + A_V u_{zz} + R,$$

$$v_t = -p_y + A_V v_{zz} + S.$$

Owing to the rigid lid approximation, the vertical velocity is zero at the surface. The mass conservation equation yields

$$w = \int_0^z (u_x + v_y) \mathrm{d}\zeta. \tag{1}$$

The boundary condition $wn_3 = 0$ at the bottom gives

$$0 = \int_0^h (u_x + v_y) \mathrm{d}\zeta, \tag{2}$$

in which $h = h(x, y)$ is the bottom depth. By taking the horizontal divergence of both sides of the momentum equations and integrating the result from surface to bottom (from (2) and the initial condition $\nabla \cdot \mathbf{v}_0 = 0$ it follows that the remaining left-hand side is zero), we obtain a two-dimensional Poisson equation for the surface pressure:

$$\nabla_h^2 p = \frac{1}{h} \int_0^h \left(\frac{\partial}{\partial x}(A_V u_{zz} + R) + \frac{\partial}{\partial y}(A_V v_{zz} + S) \right) \mathrm{d}z.$$

The correct boundary condition in the equivalent formulation for the pressure Poisson equation is a Neumann condition that is derived simply by applying the normal component of the momentum equations at the surface boundary $\partial\Omega_s$:

$$\frac{\partial p}{\partial n_s} = \mathbf{n}_s \cdot (A_V u_{zz} + R - u_t,\ A_V v_{zz} + S - v_t) \quad \text{at } \partial\Omega_s,$$

in which $\mathbf{n}_s = (n_1, n_2)$ is the unit outward-pointing normal at the boundary of the surface Ω_s.

Instead of the original problem, we consider from now on the equivalent problem in which the mass conservation equation is replaced by the pressure Poisson equation and the above boundary condition. The vertical velocity component w appearing in the terms R and S is by definition given by (1) in this new formulation and is therefore no longer considered as being an unknown.

Besson and Laydi[9] show that the solution of the original problem exists in some Sobolev space. However, additional regularity requirements are implied by the 'equivalent' formulation. *Throughout this paper we will assume u, v, p and T to be in $C^2(\bar{\Omega})$ for $t \geqslant 0$, in which $C^2(\bar{\Omega})$ is the set of functions having all spatial derivatives of order $\leqslant 2$ continuous in Ω, with continuous extensions to $\bar{\Omega}$.*

3. THE NUMERICAL ARGORITHM

3.1. The time integration scheme

The time integration procedure is such that three time levels are involved in every time step: the time level $n+1$ for which the velocity, pressure and temperature have to be calculated, and the two time levels n and $n-1$ for which the corresponding physical quantities are known from previous iterations. The choice of which terms are considered at each of the time levels $n+1$, n and $n-1$ will be justified in the next section, in which the spatial discretization and the stability of the scheme are discussed.

Using a centred difference discretization for the time derivative operator, the time integration problem for the momentum equations, the surface pressure Poisson equation and the heat equation reads as follows. Given u^{n-1}, v^{n-1}, p^{n-1}, T^{n-1}, u^n, v^n, p^n and T^n, find u^{n+1}, v^{n+1}, p^{n+1} and T^{n+1} such that

$$u^{n+1}=u^{n-1}+2\Delta t(-p_x^{n+1}+A_V u_{zz}^{n+1}+R^n),$$

$$v^{n+1}=v^{n-1}+2\Delta t(-p_y^{n+1}+A_V v_{zz}^{n+1}+S^n),$$

$$\nabla_h^2 p^{n+1}=\frac{1}{h}\int_0^h\left(\frac{\partial}{\partial x}(A_V u_{zz}^{n+1}+R^n)+\frac{\partial}{\partial y}(A_V v_{zz}^{n+1}+S^n)\right)dz,$$

$$T^{n+1}=T^{n-1}+2\Delta t[-\nabla\cdot(T^n\mathbf{v}^{n+1})+K_H\nabla_h^2 T^{n-1}+K_V T_{zz}^{n+1}],$$

with

$$R^n=g\beta\int_0^z T_x^n d\zeta-\nabla\cdot(u^n\mathbf{v}^n)+fv^n+A_H\nabla_h^2 u^{n-1},$$

$$S^n=g\beta\int_0^z T_y^n d\zeta-\nabla\cdot(v^n\mathbf{v}^n)-fu^n+A_H\nabla_h^2 v^{n-1}.$$

The boundary conditions for u^{n+1} and v^{n+1} at the surface and bottom of the domain are either Dirichlet, Neumann or a mixture of both. Although the iteration procedure to solve the above system is valid for all sorts of linear combinations of these boundary conditions, to fix ideas, let us put a Neumann condition at the surface, expressing wind stress, and a Dirichlet condition at the bottom, expressing no slip:

$$\frac{\partial u^{n+1}}{\partial z}=\sigma \quad \text{and} \quad \frac{\partial v^{n+1}}{\partial z}=\tau \quad \text{at } z=0,$$

$$u^{n+1}=v^{n+1}=0 \quad \text{at } z=h.$$

The boundary condition for the surface pressure is

$$\frac{\partial p^{n+1}}{\partial \mathbf{n}_s}=\mathbf{n}_s\cdot\mathbf{b}^n \quad \text{at } \partial\Omega_s,$$

with

$$\mathbf{b}^n=\left(A_V u_{zz}^n+R^n-\frac{u^n-u^{n-1}}{\Delta t},\ A_V v_{zz}^n+S^n-\frac{v^n-v^{n-1}}{\Delta t}\right).$$

u^{n+1}, v^{n+1} and p^{n+1} are determined using an iterative procedure. After these quantities are known, the temperature is updated with the discretized heat equation in a straightforward manner and the time step is completed. Define the following quantities:

$$\alpha^2=(2\Delta t A_V)^{-1}, \qquad \bar{p}^{n+1}=2\Delta t p^{n+1},$$

$$\bar{u}^{n+1}=u^{n+1}+2\Delta t p_x^{n+1}, \qquad \bar{v}^{n+1}=v^{n+1}+2\Delta t p_y^{n+1},$$

$$\bar{R}^n=u^{n-1}+2\Delta t R, \qquad \bar{S}^n=v^{n-1}+2\Delta t S^n.$$

Let m denote the iteration parameter of the procedure. By using the identity

$$\int^h (u_x^{n-1}+v_y^{n-1})dz=0$$

and by replacing the original quantities by the overbar quantities in the time integration scheme, we obtain the following iteration procedure:

$$\bar{u}^{n+1,m+1} - \alpha^{-2}\,\bar{u}_{zz}^{n+1,m+1} = \bar{R}^n,$$

$$\bar{v}^{n+1,m+1} - \alpha^{-2}\,\bar{v}_{zz}^{n+1,m+1} = \bar{S}^n,$$

$$\nabla_h^2 \bar{p}^{n+1,m+1} = \frac{1}{h}\int_0^h (\bar{u}_x^{n+1,m+1} + \bar{v}_y^{n+1,m+1})\,dz,$$

with boundary conditions

$$\frac{\partial \bar{u}^{n+1,m+1}}{\partial z} = \sigma \quad \text{and} \quad \frac{\partial \bar{v}^{n+1,m+1}}{\partial z} = \tau \quad \text{at } z = 0,$$

$$\bar{u}^{n+1,m+1} = -\bar{p}_x^{n+1,m} \quad \text{and} \quad \bar{v}^{n+1,m+1} = -\bar{p}_y^{n+1,m} \quad \text{at } z = h,$$

$$\frac{\partial \bar{p}^{n+1,m+1}}{\partial \mathbf{n}_s} = 2\Delta t\,\mathbf{n}_s \cdot \mathbf{b}^n \quad \text{at } \partial\Omega_s.$$

The procedure is initialized with any start value for $\bar{p}^{n+1,0}$ (e.g. $\bar{p}^{n+1,0} = 2\Delta t p^n$). Notice that in the overbar system the pressure gradient terms no longer occur in the first two equations, but instead in the corresponding Dirichlet boundary conditions. Successive iterations are coupled only through these terms, i.e. through $\bar{p}_x^{n+1,m}$ and $\bar{p}_y^{n+1,m}$. When both the surface and bottom boundary conditions are Neumann, there is no coupling and the system is solved in one iteration only. Then we get.

$$p^{n+1} = \frac{1}{2\Delta t}\,\bar{p}^{n+1,1}$$

$$u^{n+1} = \bar{u}^{n+1,1} - 2\Delta t p_x^{n+1}, \qquad v^{n+1} = \bar{v}^{n+1,1} - 2\Delta t p_y^{n+1}.$$

In this case the method is equivalent to the predictor–corrector method used in SOMS (apart from the explicit treatment of the Coriolis terms). In the case of Dirichlet conditions at the surface and/or bottom, in general more iterations are required to approximate $\bar{u}^{n+1}$, $\bar{v}^{n+1}$ and $\bar{p}^{n+1}$ properly. The condition for convergence of the procedure is derived in Section 4.1. Assuming this condition is satisfied, we formally have

$$(\bar{u}^{n+1}, \bar{v}^{n+1}, \bar{p}^{n+1}) = \lim_{m\to\infty} (\bar{u}^{n+1,m}, \bar{v}^{n+1,m}, \bar{p}^{n+1,m}).$$

All three equations in the iteration scheme are decoupled and are evaluated one after another. The first two equations are solved either directly or with a relaxation method. Since the unknowns $\bar{u}^{n+1,m+1}$ and $\bar{v}^{n+1,m+1}$ only have derivatives to z, the equations can evaluated at successive horizontal positions. The two-dimensional Poisson equation that determines $\bar{p}^{n+1,m+1}$ is solved using an SOR method or by a fast marching method for elliptic problems such as developed by Dietrich *et al.*[10] and Roache.[11]

3.2. The finite differences in space

Spatial discretization is based on the concept of control volumes.[12] The control volume approach to deriving finite difference equations relates the time derivatives of volume-integrated conserved quantity densities in each control volume to corresponding fluxes integrated over the boundaries of the control volume. The bounded domain Ω is discretized into a finite number of

cubic control volumes. The grid is regular but stretched in the vertical co-ordinate in order to model accurately the surface and/or bottom boundary layer. The control volumes for the different physical quantities are actually staggered in space, corresponding to the Arakawa-C grid[13] (see Figure 1). In this grid four different types of control volumes are distinguished: the *P*-cells for pressure, and the *U*-, *V*- and *W*-cells for the respective velocity components. Control volumes for any scalar quantity, such as temperature, coincide with the *P*-cells. The *P*-cells are considered to be either wholly inside or wholly outside Ω.

Let us derive the finite difference formulation for the heat equation to illustrate the above approach. Rewrite the heat equation in the form

$$T_t + \nabla \cdot \mathbf{F} = 0,$$

in which

$$\mathbf{F} = (uT - K_H T_x,\ vT - K_H T_y,\ wT - K_V T_z)$$

is the flux density vector. Now consider the flux through the right side of the *P*-cell i, j, k. The finite difference formulation of the flux through this side per unit time per unit area is given by

$$F_1(i+1/2, j, k) = \frac{u^n_{i,j,k}(T^n_{i+1,j,k} + T^n_{i,j,k})}{2} - K_H \frac{T^{n-1}_{i+1,j,k} - T^{n-1}_{i,j,k}}{\Delta x}.$$

Discretization of the fluxes through the other sides of the control volume is analogous, except for the implicit treatment of the vertical diffusion term in the third component of the flux density vector. The change in temperature is now calculated from the sum of the fluxes over the sides of the volume element, giving the finite difference analogue of the heat equation

$$\frac{T^{n+1}_{i,k} - T^{n-1}_{i,k}}{2\Delta t} + U\frac{T^n_{i+1,k} - T^n_{i-1,k}}{2\Delta x} + W\frac{T^n_{i,k+1} - T^n_{i,k-1}}{2\Delta z}$$
$$= K_H \frac{T^{n-1}_{i+1,k} - 2T^{n-1}_{i,k} + T^{n-1}_{i-1,k}}{(\Delta x)^2} + K_V \frac{T^{n+1}_{i,k+1} - 2T^{n+1}_{i,k} + T^{n+1}_{i,k-1}}{(\Delta z)^2}, \tag{3}$$

in which the y-component has been suppressed and the advection velocity is taken constant in order to keep the expression conveniently arranged. It is observed that the finite difference formulation for the heat equation is basically a combination of the leapfrog method for the advection terms, the forward-differencing method for the horizontal diffusion terms and implicit treatment of the vertical diffusion terms.

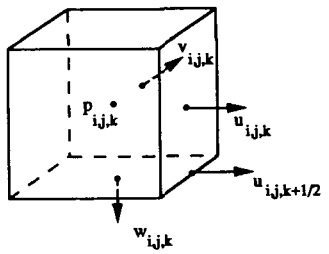

Figure 1. Arakawa-C pressure control volume. $u_{i,j,k}$, $v_{i,j,k}$ and $w_{i,j,k}$ are drawn at the centres of the *U*-, *V*- and *W*-control volumes respectively. $u_{i,j,k+1/2}$ is drawn to illustrate the notation convention

The corresponding terms in the momentum equations are treated in exactly the same manner as in the heat equation. However, the first momentum equation is evaluated with respect to the U-cells and the second with respect to the V-cells.

4. ANALYSIS OF THE ALGORITHM

4.1. Convergence of the iteration procedure

Consider the iteration procedure within a time step defined in Section 3.1. By suppressing the time step superscript we get

$$\bar{u}^{m+1} - \alpha^{-2}\,\bar{u}_{zz}^{m+1} = \bar{R},$$

$$\bar{v}^{m+1} - \alpha^{-2}\,\bar{v}_{zz}^{m+1} = \bar{S},$$

$$\nabla_h^2 \bar{p}^{m+1} = \frac{1}{h}\int_0^h (\bar{u}_x^{m+1} + \bar{v}_y^{m+1})\,dz,$$

with boundary conditions

$$\frac{\partial \bar{u}^{m+1}}{\partial z} = \sigma \quad \text{and} \quad \frac{\partial \bar{v}^{m+1}}{\partial z} = \tau \quad \text{at } z = 0,$$

$$\bar{u}^{m+1} = -\bar{p}_x^m \quad \text{and} \quad \bar{v}^{m+1} = -\bar{p}_y^m \quad \text{at } z = h,$$

$$\frac{\partial \bar{p}^{m+1}}{\partial \mathbf{n}_s} = 2\Delta t\, \mathbf{n}_s \cdot \mathbf{b}^n \quad \text{at } \partial\Omega_s.$$

Let $\bar{p}^0$ be any given start value. Although $\bar{u}^0$ and $\bar{v}^0$ are not defined by the procedure, we introduce them artificially and set $\bar{u}^0 = \bar{v}^0 = 0$. First $\bar{u}^1$, $\bar{v}^1$ and $\bar{p}^1$ are determined from the above equations. Define

$$d^m = \bar{u}^m - \bar{u}^{m-1}, \quad e^m = \bar{v}^m - \bar{v}^{m-1} \quad \text{and} \quad q^m = \bar{p}^m - \bar{p}^{m-1} \quad \text{for } m = 1,2,\ldots.$$

Notice that $d^1 = \bar{u}^1$, $e^1 = \bar{v}^1$ and $q^1 = \bar{p}^1 - \bar{p}^0$. For $m = 1, 2, \ldots$ we obtain the following system for the differences:

$$d^{m+1} - \alpha^{-2} d_{zz}^{m+1} = 0,$$

$$e^{m+1} - \alpha^{-2} e_{zz}^{m+1} = 0,$$

$$\nabla_h^2 q^{m+1} = \frac{1}{h}\int_0^h (d_x^{m+1} + e_y^{m+1})\,dz,$$

with boundary conditions

$$\frac{\partial d^{m+1}}{\partial z} = 0 \quad \text{and} \quad \frac{\partial e^{m+1}}{\partial z} = 0 \quad \text{at } z = 0,$$

$$d^{m+1} = -q_x^m \quad \text{and} \quad e^{m+1} = -q_y^m \quad \text{at } z = h,$$

$$\frac{\partial q^{m+1}}{\partial \mathbf{n}_s} = 0 \quad \text{at } \partial\Omega_s.$$

From the first two equations and the corresponding boundary conditions it follows that

$$d^{m+1} = -q_x^m \frac{\cosh(\alpha z)}{\cosh(\alpha h)}, \qquad e^{m+1} = -q_y^m \frac{\cosh(\alpha z)}{\cosh(\alpha h)}.$$

Taking the horizontal divergence $\nabla_h \cdot (d^{m+1}, e^{m+1})$ of the above expressions and putting the result in the Poisson equation yields

$$\nabla_h^2 q^{m+1} = \frac{-1}{\alpha h} \tanh(\alpha h) \left[\nabla_h^2 q^m + \alpha(\nabla_h q^m) \cdot (\nabla_h h) \tanh(\alpha h)\right].$$

Let $|\cdot|_\infty$ be the supremum norm defined on Ω_s. Since $\partial q^m / \partial \mathbf{n}_s = 0$ at $\partial\Omega_s$, we have

$$|\nabla_h q^m|_\infty \leqslant C |\nabla_h^2 q^m|_\infty,$$

in which C is a constant determined by the dimension of Ω_s. Assuming that h is sufficiently regular, define

$$\omega = \left| \frac{\tanh(\alpha h)}{\alpha h} \right|_\infty + C \left| \tanh^2(\alpha h) \frac{\nabla_h h}{h} \right|_\infty.$$

We find

$$|\nabla_h q^{m+1}|_\infty \leqslant C |\nabla_h^2 q^{m+1}|_\infty \leqslant C\omega |\nabla_h^2 q^m|_\infty \leqslant C\omega^m |\nabla_h^2 q^1|_\infty.$$

For $\omega < 1$ the method converges. Since we have put $\bar{u}^0 = \bar{v}^0 = 0$, we get

$$\bar{u} = \sum_{m=1}^{\infty} d^m, \qquad \bar{v} = \sum_{m=1}^{\infty} e^m.$$

When both the surface and bottom boundary conditions are Dirichlet, a similar condition to that above can be derived. We then get the condition

$$2 \left| \frac{\cosh(\alpha h) - 1}{\alpha h \sinh(\alpha h)} \right|_\infty + C \left| \left(\frac{\cosh(\alpha h) - 1}{\sinh(\alpha h)} \right)^2 \frac{\nabla_h h}{h} \right|_\infty < 1.$$

In all but extreme cases with very pronounced topography, both conditions are always fulfilled.

4.2. Stability

Although we do not derive a rigorous stability criterion that applies to the entire integration scheme, by considering the separate terms of interest, limitations for the time step are derived which in general are sufficient to assure stability of the method.

In order to derive the time step limitation due to the explicit treatment of the Coriolis term, consider only the corresponding part in the iteration scheme

$$u_{i,j}^{n+1} = u_{i,j}^{n-1} + 2\Delta t f \tfrac{1}{4}(v_{i,j}^n + v_{i+1,j}^n + v_{i,j-1}^n + v_{i+1,j-1}^n),$$

$$v_{i,j}^{n+1} = v_{i,j}^{n-1} - 2\Delta t f \tfrac{1}{4}(u_{i,j}^n + u_{i,j+1}^n + u_{i-1,j}^n + u_{i-1,j+1}^n),$$

in which the first equation is evaluated in the U-cells and the second in the V-cells. Notice that the velocity components in the Coriolis terms are interpolated to these cells. These interpolations are justified in the next subsection. Let $\hat{u}$ and $\hat{v}$ denote the spatial Fourier transforms of the velocity components u and v respectively. Taking the Fourier transform of the above equations yields

$$\hat{u}^{n+1} = \hat{u}^{n-1} + \gamma \hat{v}^n,$$

$$\hat{v}^{n+1} = \hat{v}^{n-1} - \bar{\gamma} \hat{u}^n,$$

with

$$\gamma=\tfrac{1}{2}\Delta t f(1+e^{-i\xi}+e^{i\theta}+e^{-i\xi}e^{i\theta}),$$

in which ξ and θ are the phase angles of the Fourier components in the x- and y-co-ordinate respectively and $\bar{\gamma}$ is the complex conjugate of γ. Define $\hat{r}^{n+1}=\hat{u}^n$ and $\hat{s}^{n+1}=\hat{v}^n$. With these definitions the above system can be written as

$$\begin{bmatrix} \hat{u}^{n+1} \\ \hat{v}^{n+1} \\ \hat{r}^{n+1} \\ \hat{s}^{n+1} \end{bmatrix} = \begin{bmatrix} 0 & \gamma & 1 & 0 \\ -\bar{\gamma} & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{u}^n \\ \hat{v}^n \\ \hat{r}^n \\ \hat{s}^n \end{bmatrix}.$$

A necessary and sufficient condition for the above scheme to be stable is that the norm of the above matrix S is smaller or equal to one, i.e. $\|\mathbf{S}\| \leqslant 1$. By determining the eigenvalues of S, it is readily verified that this is the case if $|\gamma| \leqslant 2$. The corresponding limitation for Δt follows from the definition of γ:

$$\Delta t \leqslant 1/f.$$

In a similar fashion a criterion for stability is derived with respect to the advection and diffusion terms by taking the Fourier transform of the discretized heat equation (3). The resulting condition for stability is

$$\left(\frac{|U|}{\Delta x}+\frac{|V|}{\Delta y}+\frac{|W|}{\Delta z}\right)^2 (\Delta t)^2+4K_{\mathrm{H}}\left(\frac{1}{(\Delta x)^2}+\frac{1}{(\Delta y)^2}\right)\Delta t \leqslant 1, \tag{4}$$

in which U, V and W should be interpreted as the maximum velocities that occur in Ω. Since the corresponding terms in the momentum equations are treated in exactly the same manner, the same type of restriction can be derived for these equations. Then, in the above expression K_{H} is replaced by A_{H}.

The rigid lid boundary condition eliminates the highest-frequency surface wave components. However, high-frequency internal waves can still limit the time step.

4.3. Numerical diffusion in SOMS and PGCM

By considering the integration of the Coriolis terms in the momentum equations, diffusive aspects in SOMS and PGCM are studied as a result of interpolating physical quantities to different control volumes of the Arakawa-C grid. In order to study the effect of these interpolations, we will look at the difference between the velocities obtained by the actual algorithms and the velocities that would have been obtained when no interpolations of quantities would have been necessary to approximate them with respect to different control volumes.

In SOMS the Coriolis terms are treated implicitly. Therefore the non-interpolated scheme reads as follows:

$$u_{i,j}^{n+1}=u_{i,j}^{n-1}+2\Delta t f v_{i+1/2,j-1/2}^{n+1},$$
$$v_{i,j}^{n+1}=v_{i,j}^{n-1}-2\Delta t f u_{i-1/2,j+1/2}^{n+1}.$$

The above equations are coupled through the implicit terms and both equations relate to different types of cells, the U- and V-cells respectively. In order to solve the above system, in SOMS both

equations are first interpolated to the P-cells and the new velocity components are computed with respect to these cells. The interpolated scheme is

$$\tilde{u}^{n+1}_{i-1/2,j}=\tfrac{1}{2}(u^{n-1}_{i,j}+u^{n-1}_{i-1,j})+2\Delta t f\tilde{v}^{n+1}_{i,j-1/2},$$
$$\tilde{v}^{n+1}_{i,j-1/2}=\tfrac{1}{2}(v^{n-1}_{i,j}+v^{n-1}_{i,j-1})-2\Delta t f\tilde{u}^{n+1}_{i-1/2,j},$$

in which a tilde denotes a quantity obtained with the actual algorithm. After the above system has been solved, the new velocity components are interpolated back to their respective cells to complete the procedure:

$$\tilde{u}^{n+1}_{i,j}=\tfrac{1}{2}(\tilde{u}^{n+1}_{i-1/2,j}+\tilde{u}^{n+1}_{i+1/2,j})$$
$$\tilde{v}^{n+1}_{i,j}=\tfrac{1}{2}(\tilde{v}^{n+1}_{i,j-1/2}+\tilde{v}^{n+1}_{i,j+1/2}).$$

Consider the difference between $\tilde{u}_{i,j}$ and $u_{i,j}$. With the above equalities the difference can be expressed as

$$\tilde{u}^{n+1}_{i,j}-u^{n+1}_{i,j}=\frac{(\Delta x)^2}{4}\frac{u^{n-1}_{i+1,j}-2u^{n-1}_{i,j}+u^{n-1}_{i-1,j}}{(\Delta x)^2}+\Delta t f\frac{(\Delta x)^2}{4}\left(\frac{v^{n-1}_{i+1,j-1/2}-2v^{n-1}_{i+1/2,j-1/2}+v^{n-1}_{i,j-1/2}}{(\Delta x/2)^2}\right)$$
$$+\Delta t f\frac{(\Delta y)^2}{8}\left(\frac{v^{n-1}_{i,j}-2v^{n-1}_{i,j-1/2}+v^{n-1}_{i,j-1}}{(\Delta y/2)^2}+\frac{v^{n-1}_{i+1,j}-2v^{n-1}_{i+1,j-1/2}+v^{n-1}_{i+1,j-1}}{(\Delta y/2)^2}\right)$$
$$-4(\Delta t)^2 f\frac{v^{n+1}_{i+1/2,j-1/2}-v^{n-1}_{i+1/2,j-1/2}}{2\Delta t}-2(\Delta t f)^2(\tilde{u}^{n+1}_{i+1/2,j}+\tilde{u}^{n+1}_{i-1/2,j}).$$

Let U be the maximum velocity that occurs in Ω and let L be a characteristic length scale. The characteristic time interval of interest is $T=f^{-1}$. Assume $\Delta x\geqslant\Delta y$. The orders of magnitude of the respective terms on the right-hand side are for the first term $O[U(\Delta x)^2/L^2]$, for the second and third terms $O[U\Delta t f(\Delta x)^2/L^2]$ and for the fourth and fifth terms $O[U(\Delta t f)^2]$. Assuming $\Delta t f<1$, the dominant term is the first one. This term expresses pure numerical diffusion over the time interval Δt. Thus the numerical diffusion A_N per time unit is

$$A_N=(\Delta x)^2/4\Delta t.$$

A_N is considerable. In fact, it is even more than the maximum tolerable physical horizontal diffusion in PGCM on account of the stability analysis (4) performed in Section 4.2. Further, if Δt gets smaller, A_N gets bigger, which is disadvantageous if Δt is restricted by advection or internal wave terms.

The authors of the SOMS model were aware of dispersive errors due to the implicit treatment of the Coriolis terms. However, they preferred their method to explicit treatment for reasons which relate to the coupling of the free-stream part with the bottom boundary layer part of their model.[1]

In the PGCM scheme Coriolis is treated explicitly and the momentum equations are evaluated in their respective U- and V-cells. The non-interpolated scheme is

$$u^{n+1}_{i,j}=u^{n-1}_{i,j}+2\Delta t f v^{n}_{i+1/2,j-1/2},$$
$$v^{n+1}_{i,j}=v^{n-1}_{i,j}-2\Delta t f u^{n}_{i-1/2,j+1/2}. \tag{5}$$

Consider the first equation. $v^n_{i+1/2,j-1/2}$ is approximated by

$$\tilde{v}^n_{i+1/2,j-1/2}=\tfrac{1}{4}(v^n_{i,j}+v^n_{i+1,j}+v^n_{i+1,j-1}+v^n_{i,j-1})$$

and we solve

$$\tilde{u}^{n+1}_{i,j}=u^{n-1}_{i,j}+2\Delta t f\tilde{v}^n_{i+1/2,j-1/2}.$$

The difference between $\tilde{u}^{n+1}$ and u^{n+1} is

$$\tilde{u}_{i,j}^{n+1}-u_{i,j}^{n+1}=2\Delta t f(\tilde{v}_{i+1/2,j-1/2}^{n}-\tilde{v}_{i+1/2,j-1/2}^{n})$$
$$+\Delta t f\frac{(\Delta x)^2}{8}\left(\frac{v_{i+1,j}^{n}-2v_{i+1/2,j}^{n}+v_{i,j}^{n}}{(\Delta x/2)^2}+\frac{v_{i+1,j-1}^{n}-2v_{i+1/2,j-1}^{n}+v_{i,j-1}^{n}}{(\Delta x/2)^2}\right)$$
$$+\Delta t f\frac{(\Delta y)^2}{4}\left(\frac{v_{i+1/2,j}^{n}-2v_{i+1/2,j-1/2}^{n}+v_{i+1/2,j-1}^{n}}{(\Delta y/2)^2}\right),$$

so that

$$|\tilde{u}_{i,j}^{n+1}-u_{i,j}^{n+1}|=O\left(\Delta t f\frac{(\Delta x)^2}{L^2}U\right).$$

This is the order of magnitude of the difference after one time step. The order of magnitude of the cumulative error after a characteristic time interval $T=f^{-1}$ is $O[U(\Delta x)^2/L^2]$.

5. CONCLUDING REMARKS

The equivalent formulation for the incompressible Navier–Stokes equations applied to the case in which the hydrostatic approximation is used is attractive for numerical treatment. The described PGCM numerical method for solving the equivalent system has a number of advantages. The iteration method within every time step, which is a generalized form of the predictor–corrector approach in SOMS, can be used for both Dirichlet and Neuman boundary conditions at the surface and bottom of the domain. It allows one to evaluate the surface pressure and the vertical diffusion implicitly and assures that the velocity field is solenoidal at all times. Explicit treatment of the Coriolis term decouples the momentum equations in the implicit terms and therefore avoids unwanted interpolations to different cells of the Arakawa-C grid, as in SOMS. This reduces the number of computational operations per time step and, above all, avoids numerical diffusion.

The three-dimensional PROSPER general circulation model is characterized by: its exact conservation of mass in every time step; the speed of its algorithm, which permits a high-resolution grid; and lack of numerical diffusion.

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REFERENCES

1. D. E. Dietrich, M. G. Marietta and P. J. Roache, 'An ocean modelling system with turbulent boundary layers and topography: numerical description', *Int. j. numer. methods fluids*, **7**, 833–855 (1987).
2. K. Bryan. 'A numerical method for the study of the circulation of the world ocean', *J. Comput. Phys.*, **4**, 347–376 (1969).
3. A. J. Semptner, 'An oceanic general circulation model with bottom topography', *NTIS PB 276 840*, 1976.
4. E. A. H. Zuur and D. E. Dietrich, 'The SOMS model and its application to Lake Neuchâtel', *Aquatic Sci.*, **52**, (2), 115–129 (1990).

5. A. E. Gill, *Atmosphere–Ocean Dynamics*, Academic Press, New York 1982.
6. J. Pedlosky, *Geophysical Fluid Dynamics*, Springer, New York, 1987.
7. P. M. Gresho and R. L. Sani, 'On pressure boundary conditions for the incompressible Navier–Stokes equations', *Int. j. numer methods fluids*, **7**, 1111–1145 (1987).
8. S. Abdallah and J. Dreyer, 'Dirichlet boundary conditions for the pressure Poisson equation of incompressible flow', *Int. j. numer. methods fluids*, **8**, 1029–1036 (1988).
9. O. Besson and R. Laydi, 'Some estimates for the hydrostatic approximation, and for the anisotropic Navier–Stokes equations', RAIRO, M^2AN (in press).
10. D. E. Dietrich, B. E. McDonald and A. Warn-Varnas, 'Optimized block-implicit relaxation', *J. Comput. Phys.*, **18**, 421–439 (1975).
11. P. J. Roache, 'Marching methods for elliptic problems, Parts 1–3', *Numer. Heat Transfer*, **1**, 1–25, 163–181, 183–201 (1978).
12. P. J. Roache, *Computational Fluid Dynamics*, Hermosa Publishers, 1982.
13. A. Arakawa and V. R. Lamb, 'Computational design of the basic dynamical processes of the UCLA general circulation model', *Methods in Computational Physics, Vol. 17*, Academic Press, New York, 1977, pp. 174–265.